$$u_n'' + \frac{1}{r} u_n' - \left(1 + \frac{4\pi^2 n^2}{\psi^2 r^2}\right) u_n = -A_n r, \; u_0'(r_0) = 1 + A_0, \; u_k'(r_0) = A_k$$
$$A_0 = 2 / \psi, \; A_k = (-1)^{k+1} \psi / (\pi^2 k^2 - \psi^2 / 4), \; k \geqslant 1$$

where A_n are specified coefficients of expansion of function $\operatorname{ctg}(\psi / 2) \cos\theta + \sin\theta$.

From this we obtain for $u_n(r)$ formulas in terms of cylindrical functions.

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AXISYMMETRIC PROBLEM OF THE PENETRATION OF A THIN, RIGID, SMOOTH PILE OF FINITE LENGTH INTO AN ELASTIC HALF-SPACE

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The solution of the problem in the title is given in quadratures.

When angular points (for example, a pile with a conucal tip) are present at the section occupied by the pile, tensile stresses are possible near its endpoint if it is assumed that adhesion without friction holds on this section. Otherwise cracks must be taken into account. It has been established that the stresses on the boundary of an axisymmetric pile differ from the corresponding stresses in the plane problem of wedging. Especially simple formulas are obtained in the problem of penetration of semi-infinite pile into an elastic space.

1. Plane problem. The solution of the plane problem of wedging by a thin, rigid, smooth wedge along the ox-axis of an elastic half-space is given in [1]. Let us indicate the results referred here by starting from the representation of the solution as [2]

$$2\mu \begin{Bmatrix} u \\ v \end{Bmatrix} = \operatorname{Re}\left[k^{\pm}\Phi \pm iy\Phi' + k^{\mp}\Psi \mp x\Psi'\right] \begin{Bmatrix} 1 \\ -i \end{Bmatrix} \tag{1.1}$$

$$k^{+} = k_0, \quad k^{-} = 1 + k_0, \quad k_0 = \frac{\mu}{\lambda + \mu}$$

The analytic functions of the complex argument $z = x + iy$ for the problem under consideration are

$$\Phi(z) = -\frac{q_0}{\pi}\int_L v_x' \ln\frac{z-x}{z+x}\,dx, \quad \Psi(z) = \frac{2q_0}{\pi}\int_L \frac{xv_x'}{x+z}\,dx, \quad q_0 = \frac{2\mu}{1+k_0}$$

Here the principal values are understood for the logarithms, L is the portion of the ox-axis where the derivative v_x' is not zero, and $v(x)$ is the displacement of points of the ox-axis caused by the thin wedge. Correspondingly we derive

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} = \operatorname{Re}\left[\Phi' \pm iy\Phi'' + \Psi' \mp x\Psi''\right], \quad \tau_{xy} = -\operatorname{Re}\left[y\Phi'' + ix\Psi''\right]$$

2. Wedging of a half-space. To solve the appropriate three-dimensional problem, let us use the representation of the solution given in [3]:

$$u = \langle \alpha u_0 - \beta u_3 \rangle, \quad v = \langle \beta u_0 - \alpha u_3 \rangle$$

$$w = \langle w_0 \rangle, \quad \alpha = \cos\theta, \quad \beta = \sin\theta$$

Here and henceforth, the angular brackets will denote integration with respect to θ between 0 and 2π. In the case of an isotropic body the functions $u_0(\xi, z, \theta)$, $w_0(\xi, z, \theta)$, $\xi = \alpha x + \beta y$ are the solutions of the equilibrium equations of plane elasticity theory in the ξz-plane, and the function $u_0(\xi, z, \theta)$ makes the two-dimensional Laplace operator vanish in this same plane.

If u_0, w_0 depends only on ,, z, then in the absence of torsion ($u_3 \equiv 0$), we obtain a solution possessing axial symmetry

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} \frac{1}{\rho}\langle \alpha u_0(\rho\alpha, z)\rangle, \quad w = \langle w_0(\rho\alpha, z)\rangle, \quad \rho^2 = x^2 + y^2 \tag{2.1}$$

Let a thin rigid pile of given shape

$$u_\rho(0, z) = f(z), \quad 0 \leqslant z \leqslant H$$

be driven to a depth H along the oz-axis into a half-space $z \geqslant 0$, whose boundary $z = 0$ is stress-free.

The function $f(z)$ is continuous and has a piecewise-continuous first derivative. It is required to find the state of stress and strain of the half-space. It is assumed that the desired stresses vanish at infinity and the elastic displacements are bounded everywhere.

We obtain the solution of the problem posed by rotating the solution (1.1) around the oz-axis, i.e. by setting in (2.1)

$$2\mu \begin{Bmatrix} u_0 \\ w_0 \end{Bmatrix} = \operatorname{Re}\left[k^{\mp}\Phi_0 \mp i\xi\Phi_0' + k^{\pm}\psi_0 \pm z\psi_0'\right] \begin{Bmatrix} -i \\ 1 \end{Bmatrix} \tag{2.2}$$

Here Φ_0, Ψ_0 depend on $\Omega = z + i\xi$, $\xi = \rho\alpha$ and are defined by the formulas

$$\begin{Bmatrix}\Phi_0\\ \Psi_0\end{Bmatrix} = \int\limits_L f'(\eta)\begin{Bmatrix}\Phi_{00}\\ \Psi_{00}\end{Bmatrix} d\eta, \quad \Phi_{00} = -\frac{q_0}{4\pi}\ln\frac{\Omega-\eta}{\Omega+\eta} \tag{2.3}$$

$$\Psi_{00} = \frac{q_0}{2\pi}\frac{\eta}{\eta+\Omega}$$

Using the general relationships between the stress components and formulas (5.3) from [3], we obtain

$$\begin{Bmatrix}\sigma_\rho\\ \sigma_\theta\end{Bmatrix} = \left\langle \sigma_\xi^\circ - 2\mu\alpha_\pm^2 \frac{\partial u_0}{\partial \xi}\right\rangle \quad \begin{pmatrix}\alpha_+ = \beta\\ \alpha_- = \alpha\end{pmatrix} \tag{2.4}$$

$$\sigma_z = \langle \sigma_z^\circ\rangle, \qquad \tau_{z\rho} = \langle \alpha \mathring{\tau}_{z\xi}\rangle$$

$$\begin{Bmatrix}\sigma_\xi^\circ\\ \sigma_z^\circ\end{Bmatrix} = \mathrm{Re}\,[\Phi_0' \mp i\xi\Phi_0'' + \Psi_0' \pm z\Psi_0'']$$

$$\mathring{\tau}_{z\xi} = -\,\mathrm{Re}\,[\xi\Phi_0'' + iz\Psi_0'']$$

The remaining stress components are easily found. In particular, $\tau_{\rho\theta} \equiv 0$ under the considered conditions. It is easy to verify that the stress components (2.4) satisfy the equilibrium equations in cyclindrical coordinates. Let us note that the solution of the problem posed is unique since the solution of the appropriate homogeneous problem, corresponding to zero boundary data and conditions at infinity, equals zero.

The relationships (2.1) – (2.3) permit writing the radial and axial displacements as

$$\begin{Bmatrix}u_\rho(\rho, z)\\ w(\rho, z)\end{Bmatrix} = \int\limits_L f'(\eta)\left\langle\begin{Bmatrix}\alpha u_{00}\\ w_{00}\end{Bmatrix}\right\rangle d\eta \tag{2.5}$$

Here u_{00}, w_{00} are related to Φ_{00}, Ψ_{00} by means of (2.2). The formulas for the stresses

$$\begin{Bmatrix}\sigma_z\\ \tau_{z\rho}\end{Bmatrix} = \int\limits_L f'(\eta)\left\langle\begin{Bmatrix}\overset{\circ\circ}{\sigma_z}\\ \overset{\circ\circ}{\tau_{z\rho}}\end{Bmatrix}\right\rangle d\eta \tag{2.6}$$

are written analogously.

The connection between $\sigma_z^{\circ\circ}$, $\tau_{z\rho}^{\circ\circ}$ and Φ_{00}, Ψ_{00} is determined by (2.4). The inner integrals in (2.6) and in the formulas for the other stress tensor components are evaluated in terms of elementary functions. Some properties of the solutions are established directly by using the reparesntations (2.5) and (2.6). We show that the solution constructed satisfies all the conditions of the problem posed. It is easy to verify that $\sigma_z = \tau_{z\rho} = 0$ for $z = 0$. Furthermore, let us examine the values of $\tau_{z\rho}$ and u_ρ on the oz-axis. We have

$$\tau_{z\rho} = -\int\limits_L f'(\eta)\int\limits_0^{\pi/2} \mathrm{Re}\, iz\Psi_{00}''\,\alpha d\theta d\eta = 0$$

because Ψ'''_{00} is real for $\rho = 0$. Since Ψ_{00}, Ψ_{00}' are hence also real, then

$$u_\rho(0, z) = -\frac{1}{q_0}\int\limits_L f'(\eta)\int\limits_0^{\pi/2} \mathrm{Re}\Phi^+ i\alpha d\theta d\eta$$

Here Φ_{00}^+ is the limit value of the function Φ_{00} on the upper edge of the slit $(0, \eta)$ and on the real oz-axis of the εz-plane for $z > \eta$. Taking into account the selection of the branches of the logarithms, we have

$$\operatorname{Re} i\Phi_{00}^{+} = \begin{cases} q_0\pi, & \eta > z \\ 0, & \eta < z \end{cases}$$

Hence, the radial displacement on the oz -axis is $u_\rho(0, z) = 0$ for $z > H$, while doe $z < H$

$$u_\rho(0, z) = -\int_z f'(\eta)\, d\eta = f(z)$$

since $f(H) = 0$ by continuity.

It can be verified that the elastic displacements u_ρ, w and therefore the stress components also, vanish at infinity if the depth of submersion of the pile H is finite. For example, let us find the displacement $w(\rho, 0)$ on the boundary $z = 0$ of the half-space. From (2.5) we deduce

$$w(\rho, 0) = \frac{1}{q_0}\int_L f'(\eta) \int_0^{\pi/2} \operatorname{Re} \Psi_{00} d\theta\, d\eta = \int_L \frac{f'(\eta)\,\eta d\eta}{\sqrt{\rho^2 + \eta^2}}$$

If $f'(\eta)$ does not grow, then bulging of the half-space boundary will occur under the influence of the pile driven in. For example, we have for a pile of constant thickness $2h$ on the section $(0, H_1)$ with a conical tip on the section (H_1, H_2) of the oz -axis

$$w(\rho, 0) = -h\frac{H_1 + H_2}{\sqrt{\rho^2 + H_1^2} + \sqrt{\rho^2 + H_2^2}}$$

Evaluating the inner integrals in (2.4), we obtain

$$\sigma_\rho = \frac{q_0}{2}\int_L f'(\eta)\{[z_1|z_1|^{-1}R_1 - R_2 - 2\eta z_2 R_2^3]\,k_1 - [z_1R_1R_1^* -$$

$$z_2R_2R_2^* + 2\eta(z_2R_2^3 - R_2R_2^*)]\,k_0 - 2(z_1R_1R_1^* - z_2R_2R_2^*) +$$

$$z_1|z_1|R_1^3 - z_2R_2^3\}\,d\eta$$

$$\sigma_\theta = \frac{q_0}{2}\int_L f'(\eta)\{2(R_2R_2^* - z_1|z_1|^{-1}R_1^* - \eta^2R_2^3) +$$

$$[z_1R_1R_1^* - z_2R_2R_2^* + 2\eta(z_2R_2^3 - R_2R_2^*)]\,k_0\}\,d\eta$$

$$\sigma_z = -\frac{q_0}{2}\int_L f'(\eta)\,[z_1(|z_1|R_1^3 - z_2R_2^3) + 2\eta z(2z_2^2 - \rho^2)R_2^5]\,d\eta$$

$$\tau_{z\rho} = -\frac{q_0}{2}\int_L f'(\eta)\,[|z_1|R_1^3 - z_2R_2^3 + R_1R_1^* - R_2R_2^* + 6\eta z z_2 R_2]\,d\eta$$

$$z_1 = z - \eta, \quad z_2 = z + \eta, \quad R_j^{-2} = \rho^2 + z_j^2$$

$$\frac{1}{R_j^*} = \frac{1}{R_j} + |z_j|, \quad j = 1, 2, \quad k_1 = \frac{\lambda}{\lambda + \mu}$$

On the oz-axis we have

$$\sigma_\rho = \sigma_\theta = q_0 z\int_L f'(\eta)\,[(1 - k_0)z + (3 - k_0)\eta]\,z_1^{-1}z_2^{-1}\eta d\eta \tag{2.7}$$

The integral in (2.7) is taken between H and $H + l$ for a circular pile of constant cross section with a tip of given form $f(z)$ on the section $H < z < H + l$, $H \geqslant 0$. If $f'(z)$ does not grow on this section, then the material of the half-space is compressed on the section $(0, H)$ and stretched on the section $(H + l, \infty)$. The behavior of the material on the section $(H, H+l)$ depends on the form of the function $f(z)$. For example, we obtain for a pile with a conical tip

$$\sigma_\rho = \sigma_\theta = q_0\left(1-\frac{k_0}{2}\right)he^{-1}\times \tag{2.8}$$
$$\{\chi(\xi)-(1-\gamma)\xi[\xi^2+(1+\gamma)k_2\xi+\gamma(2k_2-1)](1+\xi)^{-2}(\gamma+\xi)^{-2}\}$$
$$\chi(\xi) = {}^1/_2\ln(1+\xi)(\gamma-\xi)(1-\xi)^{-1}(\gamma+\xi)^{-1}, \quad \gamma<\xi<1$$
$$k_2 = (3-k_0)(2-k_0)^{-1}, \quad \gamma = H(H+l)^{-1}, \qquad \xi = z(H+l)^{-1}$$

It is seen that the right side in (2.8) vanishes for $\xi=\xi_0$, $\xi_0<1$, while the stresses become tensile on the section $\xi_0<\xi<1$, which corresponds to the condition of adhesion without friction. If it is absent, then a crack originates here which must be taken into account for a more accurate description of the behavior of the half-space under the influence of the pile with a conical tip. The left end of the crack cannot be located to the right of $\xi=\xi_0$.

Letting $l\to 0$, $\gamma\to 1$ in (2.8), we obtain results referring to a circular cylindrical pile of radius h, driven to a depth H

$$\sigma_p = \sigma_\theta = -q_0hH^{-1}\xi[(1-k_0)\xi+(3-k_0)](1-\xi)^{-1}(1+\xi)^{-3} \tag{2.9}$$

It is seen from (2.8), (2.9) that the stresses on the boundary of the pile differ from the corresponding stresses in the plane wedging problem [1].

If a circular pile of radius h has a tip in the form of an ellipsoid of revolution, then the integral in (2.7) is evaluated in terms of elementary functions. The formulas so obtained are awkward, hence, we limit ourselves to writing the result in the form (h, l are the semi-axes of the ellipsoid)

$$\sigma_\rho = \sigma_\theta = -q_0\frac{h}{l}\int_0^l \frac{\xi}{\sqrt{l^2-\xi^2}}\left\{\frac{z\xi}{(\xi+z_2^\circ)}+\frac{2-k_0}{4}\times\right.$$
$$\left.\left[\frac{1}{\xi-z_1^\circ}-\frac{1}{\xi+z_2^\circ}+\frac{2z}{(\xi+z_2^\circ)^2}\right]\right\}d\xi$$
$$\xi=\eta-H, \quad z_1^\circ = z-H, \quad z_2^\circ = z+H, \quad z>H$$

We have

$$\int_0^l \frac{\xi}{\sqrt{l^2-\xi^2}}\left[\frac{1}{\xi-z_1^\circ}-\frac{1}{\xi+z_2^\circ}\right]d\xi =$$
$$\frac{1}{2}(1-\tau_1^2)\tau_1^{-1}\ln(1+\tau_1)(1-\tau_1)^{-1}\tau_2^{-1}\operatorname{arctg}\tau_2$$
$$\tau_j^2 = (l-z_j^\circ)(l+z_j^\circ)^{-1}, \qquad j=1,2$$

Therefore, the stresses σ_ρ, σ_θ on the section of the oz-axis, where the pile is located, are bounded everywhere including the ends of the tip. The material on a section of the pile is compressed, while it is stretched on the oz-axis outside the section. However, the tensile stresses σ_ρ, σ_θ become unbounded as the end of the pile is approached from the right along the oz-axis.

3. Penetration of a semi-infinite pile. If a thin, smooth, circular pile of the form

$$f(z)=\begin{cases} h, & -\infty<z<0 \\ f(z), & 0\leqslant z\leqslant l \end{cases} \qquad (f(0)=h,\ f(l)=0)$$

is driven into an infinite elastic space, then we obtain the appropriate results from the above by setting $\eta=\xi+H$ and letting H tend to infinity. For example, we deduce from (2.7) on the oz-axis

$$\sigma_\rho=\sigma_\theta=q_0\frac{2-k_0}{4}I, \qquad I=\int_0^l\frac{f'(\xi)\,d\xi}{\xi-z} \tag{3.1}$$

In the case of a conical tip

$$\sigma_\rho=\sigma_\theta=-q_0\frac{h}{l}\frac{2-k_0}{4}\ln\frac{l-z}{z} \tag{3.2}$$

The material is everywhere compressed on a section of the pile, with the exception of the section $z_0<z<l$, $z_0={}^1/_2 l$.

Letting l tend to zero in (3.2), we obtain the stresses on the boundary of a thin, semi-infinite cylindrical pile

$$\sigma_\rho=\sigma_\theta=\frac{2-k_0}{4}q_0\frac{h}{z}$$

Here, in the presence of a crack at the end of the pile, we find the function governing its shape by inverting the integral $I=0$.

In conclusion, let us note that if the stress $\sigma_\rho(0,z)$ on its surface is reduced by using the thinness of the pile, then we generally obtain an unequilibrated load, whose resultant directed along the oz-axis will be

$$R=-2\pi\int_L\sigma_\rho(0,z)\,ff'dz$$

Here L is the section of the oz-axis where the integrand differes from zero. Such a force, but of opposite direction, must be applied to a smooth, thin pile to maintain it in a given position. For example, we have

$$\sigma_\rho=-q_0\frac{2-k_0}{8}\frac{h}{l}\left[\pi+\frac{1-\tau}{\tau}\ln\frac{1+\tau}{1-\tau}\right], \qquad \tau=\frac{l-z}{l+z}$$

for a semi-infinite pile of constant radius h with an elliptical tip on the section $0\leqslant z\leqslant l$

Hence, the force

$$R_1=\left(1-\frac{k_0}{2}\right)q_0Shl^{-1}\chi_0, \qquad S=\pi h^2$$

$$\chi_0=\frac{\pi}{2}+8\int_1^\infty t^2(1+t^2)^{-3}\ln t\,dt$$

should be applied to the pile at infinity.

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